

Quantum currents in the Coset Space $SU(2)/U(1)$

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Abstract

We propose a rational quantum deformed nonlocal currents in the homogenous space $SU(2)_k/U(1)$, and in terms of it and a free boson field a representation for the Drinfeld currents of Yangian double at a general level $k = c$ is obtained. In the classical limit $\hbar \rightarrow 0$, the quantum nonlocal currents become $SU(2)_k$ parafermion, and the realization of Yangian double becomes the parafermion realization of $SU(2)_k$ current algebra.

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1.Introduction

It is well known that Virasoro algebra and affine Lie algebras play a central role in conformal field theories (CFT) in two dimensional spacetime[1, 2], which in quantum theory corresponds to fields of massless particles, or massive quantum field theories (QFT) at the critical points, i.e. when the correlation length becomes infinity (scale invariant).

It is an interesting problem to describe the symmetry of the QFT off the critical points. Indeed, impressive progresses have been made in this direction, part of which is represented

by the quantum affine algebra [3, 4], which describes the symmetries of the certain lattice models and, in massive QFT, acts as the role of affine Lie algebra in CFT. For example, the spin 1/2 XXZ chain in the region $\Delta < -1$ (off-critical) was exactly solved with the help of the bosonic representation of $U_q(\widehat{sl}_2)$ at level 1 [5]; in the same way, the higher spin XXZ chain cannot be exactly solved without involving the higher level representation of the quantum affine algebra $U_q(\widehat{sl}_2)$ [6]. It is known that a quantum affine Lie algebra corresponds to certain trigonometric solution of the quantum Yang-Baxter equation (QYBE); while for a rational solution of QYBE, the corresponding algebraic structure is the the quantum double of Yangian with central extension [7]. Contrary to the quantum affine case, the Yangian double with center is obtained relatively rather late [8, 9, 10] due to some technical difficulty. Before its explicit formulation was given in [8, 9, 10], the Yangian double with center was expected to be the symmetry algebra of quantum non-local currents of massive field theories [11].

Integrable massive QFTs, which are QFTs away from the critical phase, can be obtained either by perturbations of CFT [12] with relevant fields, or in terms of free field realizations, following Lykhanov [13]. In fact the free field realization is a common approach used in both massive quantum field theories and the representation theory of their dynamical symmetry algebras[14]. The free boson representations of $U_q(\widehat{sl}_2)$ with an arbitrary level have been obtained in Refs.[6, 15]. For the Yangian double with center, the free field representation of $DY_{\hbar}(sl_2)$ with level $k(\neq 0, -2)$ was constructed in [16], the level-1 and level- k representation of $DY_{\hbar}(sl_N)$ are obtained in [17] and [18], respectively. The level- k free field representation of $DY_{\hbar}(gl_N)$ is also given in [18]. However, all of these are deformations of the Wakimoto modules. Another kind of module has been known in the representation theories of the classical [19] and quantum [20, 21] affine algebras. In the case of Yangian double with center a free field realization, which corresponds to the Feigin-Fuchs representation [19] of usual affine Lie algebras in the classical limit, is given in [22]. However, different from the corresponding quantum affine case, the coset structure in that realization is absent. So, explicit expression of rational quantum currents in the coset space $SU(2)/U(1)$ is not known yet. In the classical case, the currents defined on the coset space $SU(2)/U(1)$ subject to a $SU(2)$ nonlocal currents algebra, which is also referred to as parafermion algebra [23, 24]. Naively speaking, we would expect that the rational quantum (or \hbar)-deformed parafermion are relevant to the off-critical coset WZNW model, or equivalently, the massive WZNW model in homogeneous space $SU(2)/U(1)$. The \hbar -deformed parafermion is also relevant to the $SU(2)$ Yangian double with center $DY_{\hbar}(sl_2)_k$, which plays an important role in condensed state physics [11]. In fact the relation with the $DY_{\hbar}(sl_2)_k$ is our main criterion to define the

rational coset quantum currents. One of the main properties of the massive current is nonlocality. It is interesting to study the rational quantum deformation of coset currents, which is nonlocal even in the classical case. On the other hand, to study the rational quantum deformation of the nonlocal algebra is helpful for the investigating of the quantum deformation of the chiral vertex operator in more general scheme.

The manuscript is arranged as follows. First, we briefly review the definition of the $SU(2)$ nonlocal currents in Section 2. Then we propose a nonlocal quantum currents $\Psi(u)$ and $\Psi^\dagger(u)$, and the central extension of Yangian double for the Drinfeld new realization are given in terms of these rational quantum nonlocal currents and a $U(1)$ current. In section 3, we give a bosonic representation for a nonlocal currents through two sets of bosonic fields, the operator product expansion (OPE) of this nonlocal currents satisfies the definition of the proposed quantum nonlocal currents after a Wick rotation.

2. Quantum nonlocal currents

In this section we propose a quantum nonlocal currents (QNC) $\Psi(u)$ and $\Psi^\dagger(u)$, in the classical limit $\hbar \rightarrow 0$, this nonlocal quantum currents become the nonlocal currents $\psi(z)$ and $\psi^\dagger(z)$ of the coset space $SU(2)/U(1)$ respectively. Then in terms of such quantum nonlocal currents and a quantum $U(1)$ bosonic current, we obtain a new kind of representation for the Yangian double with center in the Drinfeld new realization. Before we going to the details, we first review briefly the theory of parafermionic currents in CFT.

Parafermionic currents are primary fields of the 2d CFT. The general parafermion defined for root lattices are proposed in [24]. For the case of $SU(2)$, the affine Kac-Moody currents and parafermionic currents are related by the relations,

$$\begin{aligned}\chi_+(z) &= \sqrt{k} : \psi(z) \exp(i\phi(z)/\sqrt{k}) :, \\ \chi_-(z) &= \sqrt{k} : \psi^\dagger(z) \exp(-i\phi(z)/\sqrt{k}) :, \\ h(z) &= i\sqrt{k} \partial_z \phi(z),\end{aligned}\tag{2.1}$$

where $\chi_\pm(z)$ and $h(z)$ are currents of $SU(2)$ affine algebra, and the radial ordering of the parafermions are given by

$$R(\psi_\alpha(z)\psi_\beta(w))(z-w)^{2\alpha\beta/k} = R(\psi_\beta(w)\psi_\alpha(z))(w-z)^{2\alpha\beta/k},\tag{2.2}$$

in which $\alpha, \beta = \pm$. We will drop the R symbol in the following without of any confusion. The OPE of the parafermionic fields defined by [24]

$$\begin{aligned}
\psi_{\pm}(z)\psi_{\pm}(w)(z-w)^{2/k} &= \text{reg.} \\
\psi_{+}(z)\psi_{-}(w)(z-w)^{-2/k} &= \frac{1}{(z-w)^2} + \text{reg.},
\end{aligned} \tag{2.3}$$

For rational quantum deformation the $SU(2)$ currents algebra becomes the central extension of Yangian double for the Drinfeld new realization. From this point view we propose rational quantum deformation of the above nonlocal currents. The results are:

Proposition 1 *The rational quantum deformation of nonlocal currents for $SU(2)/U(1)$ can be defined as:*

$$\begin{aligned}
((u-v) + \hbar) \frac{\Gamma(\frac{-(u-v)}{k\hbar} + \frac{1}{k} - 1)}{\Gamma(\frac{-(u-v)}{k\hbar} - \frac{1}{k} - 1)} \Psi(u)\Psi(v) &= ((u-v) - \hbar) \frac{\Gamma(\frac{-(v-u)}{k\hbar} - \frac{1}{k} - 1)}{\Gamma(\frac{-(v-u)}{k\hbar} - \frac{1}{k} - 1)} \Psi(v)\Psi(u), \\
((u-v) - \hbar) \frac{\Gamma(\frac{-(u-v)}{k\hbar} + \frac{1}{k})}{\Gamma(\frac{-(u-v)}{k\hbar} - \frac{1}{k})} \Psi^{\dagger}(u)\Psi^{\dagger}(v) &= ((u-v) + \hbar) \frac{\Gamma(\frac{-(v-u)}{k\hbar} + \frac{1}{k})}{\Gamma(\frac{-(v-u)}{k\hbar} - \frac{1}{k})} \Psi^{\dagger}(v)\Psi^{\dagger}(u), \\
\frac{\Gamma(\frac{-(u-v)}{k\hbar} - \frac{1}{k} - \frac{1}{2})}{\Gamma(\frac{-(u-v)}{k\hbar} + \frac{1}{k} - \frac{1}{2})} \Psi(u)\Psi^{\dagger}(v) &= \frac{\Gamma(\frac{-(v-u)}{k\hbar} - \frac{1}{k} - \frac{1}{2})}{\Gamma(\frac{-(v-u)}{k\hbar} + \frac{1}{k} - \frac{1}{2})} \Psi^{\dagger}(v)\Psi(u),
\end{aligned} \tag{2.4}$$

At first glance the above relations may seem very strange. Their actual meaning in the theory of quantum deformed algebras will soon be clear if we resort to the construction of Yangian double analogous to that of $SU(2)$ current algebra in terms of ordinary parafermionic currents, and indeed, such a construction exists, and its classical limit $\hbar \rightarrow 0$ gives precisely the parafermionic construction of $SU(2)$ currents.

To make the above statements more concrete, we need to introduce another bosonic currents.

$$[\hat{c}(t), \hat{c}(t')] = \frac{\sinh \hbar t \sinh \frac{k}{2} \hbar t}{\hbar^2 t} \delta(t + t'). \tag{2.5}$$

Defining the bosonic currents

$$\begin{aligned}
C^{+}(u) &= \exp\left\{-\hbar \int_{-\infty}^0 dt e^{\frac{k}{4}\hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2}\hbar t} \hat{c}(t) - \hbar \int_0^{\infty} dt e^{-\frac{k}{4}\hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2}\hbar t} \hat{c}(t)\right\}, \\
C^{-}(u) &= \exp\left\{\hbar \int_{-\infty}^0 dt e^{-\frac{k}{4}\hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2}\hbar t} \hat{c}(t) + \hbar \int_0^{\infty} dt e^{+\frac{k}{4}\hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2}\hbar t} \hat{c}(t)\right\},
\end{aligned} \tag{2.6}$$

we have the following exchange relations,

$$\begin{aligned}
\frac{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k} - 1)}{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k} - 1)} C^+(u) C^+(v) &= C^+(v) C^+(u) \frac{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k} - 1)}{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k} - 1)}, \\
\frac{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k})}{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k})} C^-(u) C^-(v) &= C^-(v) C^-(u) \frac{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k})}{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k})}, \\
\frac{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k} + \frac{1}{2})}{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k} + \frac{1}{2})} C^+(u) C^-(v) &= C^-(v) C^+(u) \frac{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k} + \frac{1}{2})}{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k} + \frac{1}{2})}.
\end{aligned} \tag{2.7}$$

If we perform the Wick rotation for currents C^\pm , namely replacing \hbar by $-i\hbar$, and defining the quantum currents as

$$\begin{aligned}
E(u) &=: \Psi(u) C^+(u) :, \\
F(u) &=: \Psi^\dagger(u) C^-(u) :,
\end{aligned} \tag{2.8}$$

we obtain the Drinfeld currents of Yangian double with center, with relations

$$\begin{aligned}
[H^\pm(u), H^\pm(v)] &= 0, \quad [c, \text{everything}] = 0, \quad c = k, \\
(u - v + \hbar \mp \frac{k}{2}\hbar)(u - v \pm \frac{k}{2}\hbar - \hbar) H^\pm(u) H^\mp(v) \\
&= (u - v - \hbar \mp \frac{k}{2}\hbar\hbar)(u - v \pm \frac{k}{2}\hbar + \hbar) H^\mp(v) H^\pm(u), \\
(u - v + \hbar \mp \frac{k}{4}\hbar) H^\pm(u) E(v) &= (u - v - \hbar \mp \frac{k}{4}\hbar) E(v) H^\pm(u), \\
(u - v - \hbar \pm \frac{k}{4}\hbar) H^\pm(u) F(v) &= (u - v + \hbar \pm \frac{k}{4}\hbar) F(v) H^\pm(u), \\
(u - v + \hbar) E(u) E(v) &= (u - v - \hbar) E(v) E(u), \\
(u - v - \hbar) F(u) F(v) &= (u - v + \hbar) F(v) F(u), \\
[E(u), F(v)] &= \frac{1}{\hbar} \left(\delta(u - v + \frac{k}{2}\hbar) H^+(u + \frac{k}{2}\hbar) - \delta(u - v - \frac{k}{2}\hbar) H^-(v + \frac{k}{2}\hbar) \right),
\end{aligned} \tag{2.9}$$

wherein

$$\begin{aligned}
H^+(u) &= \exp\{2\hbar \int_0^{+\infty} dt e^{-iut} \hat{c}(t)\}, \\
H^-(u) &= \exp\{-2\hbar \int_{-\infty}^0 dt e^{-iut} \hat{c}(t)\}.
\end{aligned} \tag{2.10}$$

Since the fields $C^\pm(u)$ and $H^\pm(u)$ are only involved in the quantum $U(1)$ current, the coset structure of classical currents are preserved, and the coset structure could not be preserved by another kind of free field representation [22].

3. Bosonization of the quantum nonlocal currents

In order to see that the strange rational deformed quantum currents are well-defined, i.e. their definition is not empty, in this section we give a free field realization of the quantum nonlocal currents. First we introduce two kinds of Heisenberg algebras,

$$\begin{aligned} [\hat{b}(t), \hat{b}(t')] &= -\frac{\sinh \hbar t \sinh \frac{k}{2} \hbar t}{\hbar^2 t} \delta(t+t'), \\ [\hat{\lambda}(t), \hat{\lambda}(t')] &= \frac{\sinh \hbar t \sinh \frac{k+2}{2} \hbar t}{\hbar^2 t} \delta(t+t'). \end{aligned} \quad (3.1)$$

Then defining the following intermediate fields

$$\begin{aligned} B^+(u) &= \exp\left\{-\hbar \int_{-\infty}^0 dt e^{\frac{k}{4} \hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2} \hbar t} \hat{b}(t) - \hbar \int_0^{\infty} dt e^{-\frac{k}{4} \hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2} \hbar t} \hat{b}(t)\right\}, \\ B^-(u) &= \exp\left\{\hbar \int_{-\infty}^0 dt e^{-\frac{k}{4} \hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2} \hbar t} \hat{b}(t) + \hbar \int_0^{\infty} dt e^{+\frac{k}{4} \hbar t} \frac{e^{-iut}}{\sinh \frac{k}{2} \hbar t} \hat{b}(t)\right\}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \Lambda_+(u) &= \exp\left\{-2\hbar \int_0^{+\infty} dt \frac{\sinh \frac{\hbar}{2} t}{\sinh \hbar t} e^{-iut} \hat{\lambda}(t)\right\}, \\ \Lambda_-(u) &= \exp\left\{2\hbar \int_{-\infty}^0 dt \frac{\sinh \frac{\hbar}{2} t}{\sinh \hbar t} e^{-iut} \hat{\lambda}(t)\right\}, \\ \beta_+(u) &= \exp\left\{-2\hbar \int_0^{+\infty} dt \frac{\sinh \frac{\hbar}{2} t}{\sinh \hbar t} e^{-iut} \hat{b}(t)\right\}, \\ \beta_-(u) &= \exp\left\{2\hbar \int_{-\infty}^0 dt \frac{\sinh \frac{\hbar}{2} t}{\sinh \hbar t} e^{-iut} \hat{b}(t)\right\}, \end{aligned} \quad (3.3)$$

we have the following formal commutation relations, i.e. relations to be understood in the sense of analytic continuation,

$$\begin{aligned} \beta_+(u) B^\pm(v) &= \frac{i(u-v) \pm \frac{k}{4} \hbar \pm \frac{1}{2} \hbar}{i(u-v) \pm \frac{k}{4} \hbar \mp \frac{1}{2} \hbar} B^\pm(v) \beta_+(u), \\ B^\pm(u) \beta_-(v) &= \frac{i(u-v) \pm \frac{k}{4} \hbar \pm \frac{1}{2} \hbar}{i(u-v) \pm \frac{k}{4} \hbar \mp \frac{1}{2} \hbar} \beta_-(v) B^\pm(u), \\ \Lambda_+(u) \Lambda_-(v) &= \frac{\Gamma(\frac{i(u-v)}{2\hbar} + \frac{k+2}{4}) \Gamma(\frac{i(u-v)}{2\hbar} + \frac{k+6}{4})}{\Gamma(\frac{i(u-v)}{2\hbar} - \frac{k+2}{4}) \Gamma(\frac{i(u-v)}{2\hbar} - \frac{k-2}{4})} \\ &\quad \times \frac{\Gamma^2(\frac{i(u-v)}{2\hbar} - \frac{k}{4})}{\Gamma^2(\frac{i(u-v)}{2\hbar} + \frac{k+4}{4})} \Lambda_-(v) \Lambda_+(u), \end{aligned}$$

$$\begin{aligned}
\beta_+(u)\beta_-(v) &= \frac{\Gamma(\frac{i(u-v)}{2\hbar} - \frac{k}{4})\Gamma(\frac{i(u-v)}{2\hbar} - \frac{k-4}{4})}{\Gamma(\frac{i(u-v)}{2\hbar} + \frac{k}{4})\Gamma(\frac{i(u-v)}{2\hbar} + \frac{k+4}{4})} \\
&\quad \times \frac{\Gamma^2(\frac{i(u-v)}{2\hbar} + \frac{k+2}{4})}{\Gamma^2(\frac{i(u-v)}{2\hbar} - \frac{k-2}{4})}\beta_-(v)\beta_+(u),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\frac{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k} - 1)}{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k} - 1)}B^+(u)B^+(v) &= B^+(v)B^+(u)\frac{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k} - 1)}{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k} - 1)}, \\
\frac{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k})}{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k})}B^-(u)B^-(v) &= B^-(v)B^-(u)\frac{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k})}{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k})}, \\
\frac{\Gamma(\frac{i(u-v)}{k\hbar} - \frac{1}{k} + \frac{1}{2})}{\Gamma(\frac{i(u-v)}{k\hbar} + \frac{1}{k} + \frac{1}{2})}B^+(u)B^-(v) &= B^-(v)B^+(u)\frac{\Gamma(\frac{i(v-u)}{k\hbar} - \frac{1}{k} + \frac{1}{2})}{\Gamma(\frac{i(v-u)}{k\hbar} + \frac{1}{k} + \frac{1}{2})}.
\end{aligned} \tag{3.5}$$

Using these results, we can get a realization of the quantum nonlocal currents as follows,

$$\begin{aligned}
\Psi(u) &= \frac{1}{\hbar} : \{ \beta_+(u + i\frac{k+2}{4}\hbar)\Lambda_+(u + i\frac{k}{4}\hbar) - \beta_-(u - i\frac{k+2}{4}\hbar)\Lambda_-(u - i\frac{k}{4}\hbar) \} B^+(u) :, \\
\Psi^\dagger(u) &= -\frac{1}{\hbar} : \{ \beta_+(u - i\frac{k+2}{4}\hbar)\Lambda_+^{-1}(u - i\frac{k}{4}\hbar) - \beta_-(u + i\frac{k+2}{4}\hbar)\Lambda_-^{-1}(u + i\frac{k}{4}\hbar) \} B^-(u) : \tag{3.6}
\end{aligned}$$

By direct calculation, we can show that the OPE of the above nonlocal currents coincide with the defining relations (2.4).

In this paper a set of rational deformed quantum nonlocal $SU(2)$ currents are proposed, bosonization and their relation with central extension of Yangian double in the Drinfeld new realization are also discussed. A related interesting object, i.e. the screening currents for these quantum nonlocal currents, which is important for the calculation of the correlation functions, will be considered in a separate paper.

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